sailing on a south course, from Jamaica to Carthagena, sees Don Blass right before him steering west, along the shore. He now continually bears directly upon him in a right line; when coming up with him, it appears that the Don had sailed 8 leagues during the chace, and that the said admiral was 7 leagues distant from him when the chace began: Now, supposing each ship's motion to be uniform during the whole chace, to find from thence the distance sail'd by Admiral Vernon?"

The following problem was propounded by Thomas Perryam in *Gentleman's Diary*, 1749: "A Captain of a Privateer seeing a Merchant Ship at S.S.E. sailing due West, continually bears directly upon her in a right Line; and coming up with her, it appear'd that the Merchant's Ship sail'd 30 Miles during the Chace, and that the said Privateer was 21 Miles distant from her when the Chace first began: Now supposing each Ship's Motion to be uniform during the whole Chace, To find from thence the Distance sail'd by the said Privateer, its East and West Departure, and also their Difference of Latitude when they were North and South of each other." A solution by fluxions was given in 1750.

Another question on the subject was propounded by John Ash in *Ladies Diary*, 1748, (Leybourn's reprint, vol. 2, p. 15) in the following form: "A spider, at one corner of a semi-circular pane of glass, gave uniform and direct chase to a fly, moving uniformly along the curve before him; the fly was 30° from the spider at the first setting out, and was taken by him at the opposite corner. What is the ratio of both their uniform motions[1]?" On this question the editor said "Mr. Landen sent us a true method [of solution]; but the calculus being so operose, it was not wrought out. And no method appearing to us yet elegant enough for a place, it will be next year before we shall have time to catch the solution." No further remarks on it appear in subsequent numbers of the *Diary*.

It would seem then that before 1750 such questions had become sufficiently familiar in England to appear in popular journals.

W. W. ROUSE BALL.

TRINITY COLLEGE, CAMBRIDGE.
March 1, 1921.

18. Radius of the sphere circumscribing a tetrahedron. A member of the Association proposed the following problem which during the past century has been frequently solved: "Find, in terms of the lengths of the edges of a tetrahedron, the radius, R, of the circumscribing sphere." Denoting the pairs of opposite edges of the tetrahedron by a, a_1; b, b_1; c, c_1, L.N.M.Carnot derived[2] in 1806 the relation

$$\begin{aligned}4R^2(&a_1^4a^2 + a^4a_1^2 + b_1^4b^2 + b^4b_1^2 + c_1^4c^2 + c^4c_1^2 + a^2b_1^2c_1^2 + c^2a_1^2b_1^2\\ &+ b^2a_1^2c_1^2 + a^2b^2c^2 - a^2b^2b_1^2 - b^2c^2b_1^2 - b^2a_1^2b_1^2 - a^2a_1^2b_1^2 - b^2b_1^2c_1^2\\ &- c^2b_1^2c_1^2 - a^2b^2a_1^2 - a^2c^2a_1^2 - b^2c^2c_1^2 - a^2c^2c_1^2 - a^2a_1^2c_1^2 - c^2a_1^2c_1^2)\\ &+ 2b^2c^2b_1^2c_1^2 + 2a^2c^2a_1^2c_1^2 + 2a^2b^2a_1^2b_1^2 - a^4a_1^4 - b_1^4b^4 - c_1^4c^4 = 0.\end{aligned}$$

[1] Compare the notes on problem 2801 below.—EDITOR.

[2] *Mémoire sur la Relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace.* Paris, 1806, p. 11.

In 1847 Brassine gave,[1] without proof, the formula:

$$R = \frac{1}{24\cdot V}\sqrt{(aa_1+bb_1+cc_1)(aa_1+bb_1-cc_1)(aa_1+cc_1-bb_1)(cc_1+bb_1-aa_1)},$$

or

$$R = \frac{1}{6\cdot V}\sqrt{p(p-aa_1)(p-bb_1)(p-cc_1)},$$

where V is the volume of the tetrahedron, and $2p = aa_1 + bb_1 + cc_1$. That is, six times the product of the volume of the tetrahedron and of the radius of the circumscribed sphere is numerically equal to the area of a triangle whose sides are of lengths aa_1, bb_1, cc_1. This result was published in 1821 by Crelle.[2] In 1752 Euler gave in effect, the following expression,[3] in modern notation, for V:

$$\frac{1}{12}\sqrt{\begin{aligned}&[a^2a_1^2(b^2+b_1^2+c^2+c_1^2) - a^2a_1^2(a^2+a_1^2) + b^2b_1^2(c^2+c_1^2+a^2+a_1^2)\\ &\quad - b^2b_1^2(b^2+b_1^2) + c^2c_1^2(a^2+a_1^2+b^2+b_1^2) - c^2c_1^2(c^2+c_1^2)\\ &\quad - a^2b^2c^2 - a^2b_1^2c_1^2 - b^2c_1^2a_1^2 - c^2a_1^2b_1^2].\end{aligned}}$$

Killing and Hovestadt state this relation[4] in the form

$$\begin{aligned}144V^2 = {}&(a^2+b^2+c^2+a_1^2+b_1^2+c_1^2)(a^2a_1^2+b^2b_1^2+c^2c_1^2) - 2a^2a_1^2(a^2+a_1^2)\\ &- 2b^2b_1^2(b^2+b_1^2) - 2c^2c_1^2(c^2+c_1^2) - a_1^2b^2c^2 - a^2b_1^2c^2 - a^2b^2c_1^2\\ &- a_1^2b_1^2c_1^2.\end{aligned}$$

Baltzer gave[5] the determinant form:

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & a^2 & b^2 & c^2\\ 1 & a^2 & 0 & c_1^2 & b_1^2\\ 1 & b^2 & c_1^2 & 0 & a_1^2\\ 1 & c^2 & b_1^2 & a_1^2 & 0\end{vmatrix},$$

and Joachimsthal, and Dostor the following:[6]

$$576V^2R^2 = -\begin{vmatrix} 0 & a^2 & b^2 & c^2\\ a^2 & 0 & c_1^2 & b_1^2\\ b^2 & c_1^2 & 0 & a_1^2\\ c^2 & b_1^2 & a_1^2 & 0\end{vmatrix} = -\begin{vmatrix} 0 & aa_1 & bb_1 & cc_1\\ aa_1 & 0 & cc_1 & bb_1\\ bb_1 & cc_1 & 0 & aa_1\\ cc_1 & bb_1 & aa_1 & 0\end{vmatrix}.$$

[1] *Nouvelles Annales de Mathematiques*, vol. 6, 1847, p. 227; a demonstration by de Perrodil is given on pages 396–398 of the same volume.

[2] A. L. Crelle, *Sammlung mathematischer Aufsätze und Bemerkungen*, Berlin, vol. 1, p. 117.

[3] *Novi comment. acad. sc. Petrop.*, vol. 4 (1752–53), 1758, p. 159.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Leipzig, vol. 2, 1913, p. 421.

[5] R. Baltzer, *Théorie et Applications des Déterminants*. Traduit de l'allemand. Paris, 1861 p. 206.

[6] F. Joachimsthal, Crelle's *Journal*, vol. 40, 1850, p. 33. G. Dostor, *Eléments de la Théorie des, Déterminants*, deuxième éd., Paris, 1883, pp. 281–282; also *Nouvelles Annales de Mathématiques*, vol. 32, 1873, p. 374.

Other derivations of expressions for the radius of the circumscribed sphere were given by Legendre, *Eléments de Géométrie*, 8e éd., Paris, 1809, pp. 302–304 of note V; by Baltzer, *Die Elemente der Mathematik*, Leipzig, volume 2, 1860, pp. 348–349; and by G. Holzmüller, *Elemente der Stereometrie*, Leipzig, volume 2, 1900, pp. 228–231.

ARC.

PROBLEMS—SOLUTIONS.

2801 [1920, 31; 1921, 54–61, 91–97]. Proposed by A. S. HATHAWAY, Houston, Texas.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $k : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

I. Additional Remarks by the Proposer.

If we let $d\sigma$ denote differential along the apparent path of the dog, we shall have

$$\left(\frac{d\sigma}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = MQ^2/a^2,$$

whence $d\sigma/dt = MQv/a$, where $v = ds/dt$ is the real velocity of the duck.

When $k > 1$, $MQ \geqq ka - a$, and so the apparent velocity of the dog within or on the edge of the pond is never less than $(k - 1)v$.

When $k = 1$, the apparent velocity of the dog is never zero within the pond, nor on the edge except at P. In fact the dog will approach P as limiting point.

When $k < 1$, MQ will be zero at the point where the ray $\sin\theta = k$ intersects the circle CKP. Let α be this value of θ and let w be the distance of Q from this point. Then

$$w^2 = r^2 + a^2\cos^2\alpha - 2ar\cos\alpha\cos(\theta - \alpha),$$

and

$$w\,dw/ds = -k(r + a\cos\alpha)[1 - \cos(\theta - \alpha)],$$

which is never positive, and is zero only when $\theta = \alpha$. Therefore w is always decreasing (except when $\theta = \alpha$), and the apparent path of the dog is a spiral converging to the point where the apparent velocity is zero, $\theta = \alpha$, $r = a\cos\alpha$.

It is interesting to note that if the speed of the dog be less than that of the duck, there is one position of starting where the relative positions of the two will remain unchanged, and that this is the limiting goal of the dog from whatever position he starts within the pond.

A number of interesting results may be deduced by determining entrances and exits on fixed curves. Thus on $r = ma\cos\theta$, exits and entrances are divided by the ray $(m - 2)\sin\theta + k = 0$, exits over the shorter arc.

II. Remark by H. P. Manning, Providence, R. I.

Mr. Morley has made a slight mistake (*1921*, 60) in regard to the cusps and inflections of his integral curves. The substitutions on page 55 lead first to an irrational equation for dv/dp, and the second derivative of v for a curve of pursuit is zero only when the integral curve crosses that part of the cubic which lies to the left of the point for which $p = c$. Between the two parts of the cubic all integral curves are concave downwards, but at any point outside of the cubic they curve in opposite directions; and so from a cusp the two branches curve away in opposite directions, one less steep and the other steeper than the slope $-2c$, and the cusps are all ordinary cusps, and not of the rhamphoid type. Indeed, one such cusp is at the point $(0, 1)$, where one branch corresponds to the curve of pursuit and the other to the "curve of flight."

III. Remarks and Historical Notes by H. P. Manning, and R.C. Archibald, Brown University.

We have remarked before (*1921*, 92) that the earliest reference then found, to a curve of pursuit where the pursued moved on a circle, was in Ficklin's problem of 1859. Mr. Ball has noted